9.3. Primary fields, OPE, fusion

Def: Given CFT.

Φ_i : primary field if $\forall n \in \mathbb{Z}$

$$[L_n, \Phi_i(z)] = z^{n+1}\partial_z \Phi_i(z) + h_i(n+1)z^n \Phi_i(z) \quad (P)$$

!: This condition is equiv to:

1, $T(z_1)\,\Phi_i(z_2) \sim \frac{h_i}{(z_1-z_2)^2}\Phi_i(z_2) + \frac{1}{z_1-z_2}\partial_{z_2}\Phi_i(z_2)$

2, $\Phi_i(z)$ exhibits local conf covariance.

That is exponentiate the generators L_n we get:

$$U(e^{tL_n})\,\Phi_i(z)\,U(e^{-tL_n}) = \left(\frac{d\,\gamma_t}{dz}\right)^{h_i}\Phi_i(\gamma_t(z)),$$

for $U \in Vir^{grp}$-rep^u.

$W_t(z) = z + t z^{n+1}$

Indeed, apply $\frac{d}{dt}\Big|_{t=0}$ we get (P)

- $G_i = \langle \Omega | \Phi_i | \Omega \rangle$ can be thought of as meromorphic section in $K^h \otimes \bar{K}^{\bar{h}}$ for $K = (T^{1,0})^* \mathbb{C}$, $\bar{K} = (T^{0,1})^* \mathbb{C}$ (or some Σ)

Prop: (local) CWI: Φ_i:

$$\langle T(z) \prod_j^n \Phi_j(z_j) \rangle$$

$$\sum_i^n \left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_{z_i} \right) \langle \prod_j \Phi_j(z_j) \rangle$$

Prf: Use OPE and residual $\oint$

$$\oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \epsilon(w) \langle T(w) \phi_1(z_1, \bar{z}_1) \cdots \phi_N(z_N, \bar{z}_N) \rangle \qquad (\epsilon(w) = (w-z_i)^{-n+1})$$

$$= \sum_{i=1}^{N} \langle \phi_1(z_1, \bar{z}_1) \cdots \left(\oint_{\mathcal{C}(z_i)} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \epsilon(w) T(w) \phi_i(z_i, \bar{z}_i) \right) \cdots \phi_N(z_N, \bar{z}_N) \rangle$$

$$= \sum_{i=1}^{N} \langle \phi_1(z_1,\bar z_1)\cdots \left(\oint_{C(z_i)} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \epsilon(w) \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i} \right) \phi_i(z_i,\bar z_i) \right) \cdots \phi_N(z_N,\bar z_N)\rangle$$

$$= \oint_{C(0)} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \epsilon(w) \sum_{i=1}^{N} \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i} \right) \langle \phi_1(z_1,\bar z_1)\cdots \phi_N(z_N,\bar z_N)\rangle,$$

! For $\epsilon(w) \in \{1, w, w^2\}$ then we get Global CWI.

Def: Φ_i: quasiprimary $\in F(H)$

if Φ_i: satisfy (global) conf cov.

! In 2D, primary $\not\Leftarrow\Rightarrow$ quasiprimary

but $>2D$ primary $\not\Leftarrow\Rightarrow$ quasiprimary.

Def: $n \geq 0$, Φ_i: primary

$$L_n \Phi_i(z)\Omega = [L_n, \Phi_i(z)]\Omega$$

$$= z^{n+1}\partial_z \Phi(z)\Omega + h_i(n+1)z^n \Phi_i(z)\Omega.$$

Let $v \equiv \lim_{z\to 0} \Phi_i(z)\Omega$ (asymptotic state of Φ_i)

then $L_0 v = h_i v$, $L_n v = 0$ $(n>0)$

! $\mathcal{H} \supseteq \mathrm{Span}\left\{\prod_{i=1}^{K} L_{-n_i} V\right\}_{n_i \geq 0}^{K \in \mathbb{N}} \in \mathrm{Vir}\text{-mod.}$

A state $\prod_i L_{-n_i} V$: descendants of V

Def. $B_1 \equiv \{ i \in B_0 \mid \Phi_i : \text{primary} \}$.

then for $i \in B_1$

$$\left\{ \Phi_i^{\alpha}(z) = \prod_{j=1}^{N} L_{-\alpha_j}(z)\, \Phi_i(z) \right\}_{\alpha_1 \geq \dots \geq \alpha_N}^{\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N} \equiv [\Phi_i]$$

(vect space of) family of desecendants of Φ_i

where

$$L_{-n}(z) \equiv \mathrm{Res}_{\zeta}\, T(\zeta)(\zeta - z)^{-n-1} d\zeta$$

$$(L_{-n} = L_{-n}(0))$$

- $\Phi_i^{\alpha}(z)$: secondary field/descendant of $\Phi_i(z)$

Def: $\mathcal{F}(\mathcal{H}) \equiv \{ \Phi_i : \text{Fields on } \mathcal{H} \ \ i \in B_0 \}$

$\mathcal{F}_1(\mathcal{H}) \equiv \{\Phi_i : \text{primary fields}\}$.

$\mathcal{F}_2(\mathcal{H}) \equiv \{\Phi_i : \text{secondary fields}\}$.

$(\mathcal{F}_2(\mathcal{H}) \oplus \mathcal{F}_1(\mathcal{H}) = \mathcal{F}(\mathcal{H}))$

Prop (State-field correspondance):

If the asymtotic states of primary fields with their descendants generate dense $\mathcal{H} \supseteq V \in \text{Vir-rep}$ then

$\forall a \in V, \exists! \Phi \in \mathcal{F}(\mathcal{H})$ s/t

$a = \lim_{z \to 0} \Phi(z)\Omega$ and vice versa.

That is, $V \simeq \mathcal{F}(\mathcal{H}) \in \text{Vect}$)

$$\lim_{z\to 0} \Phi(z)\Omega \leftrightarrow \Phi(z)$$

$$a \mapsto Y(a,z) \quad \text{s/t.}$$

$$\lim_{z \to 0} Y(a,z)|0\rangle = a$$

this is motivation of vertex alg

Prf: Proof is clear for $\Phi_i \in F_1(H)$ since $a = \lim_{z\to 0} \Phi_i(z)\Omega$ is asymptotic state.

So check for secondary fields

$$w \equiv \prod_{j=1} L_{-\alpha_j} \Phi_i(0)\Omega$$

$$= \lim_{z\to 0} \Phi_i^{\alpha}(z)\Omega$$

! For a fixed $z \in \mathbb{C}$, $\Phi_i \in F_1(H)$

$v \equiv \Phi_i(z)$ by state-fld.

$$\text{Span}\left\{ \prod_{j=1}^{N} L_{-\alpha_j}(z) \overset{H \otimes V}{v} \right\}_{\alpha_1 \geq \dots \geq \alpha_N > 0 \in \mathbb{N}}^{N \in \mathbb{N}}$$

$\in$ Vir-rep with weight (c_i, h_i)

then $L_0 v = h_i v$

$$L_n v = \begin{cases} 0 & n > 0 \\ \Phi_i^n(z) & n < 0 \end{cases}$$

Axiom 6: $\Phi_i \in F_1(H)$ then

$\Phi_i(z_1)\Phi_j(z_2)$

$$\sum_{k\in B_0} C_{ijk}(z_1-z_2)^{h_k-h_i-h_j}\left(\Phi_k(z_2)+O(z_1-z_2)\right)$$

structural constant for 3-pt funcs.

contributions of descendants

(Operator product expansion)

"Prop" (Bootstrap): If OPE is associative then

$(h_i, c_i, C_{ijk})_{i,j,k\in B_0}$ completely determines a CFT_{2D}.

"prf:" Since a CFT from OS axioms are determined by correlation functions with OPEs we can turn any n-point func to (n-1)-point. Induct to 3-point funcs where that is completely determined by C_{ijk}.

Ex: Let's compute 4-pt func in CFT_2 using (4) with only chiral part:

$$G_{1234}(z_1,z_2,z_3,z_4) = F(r(z)) \prod_{i<j} z_{ij}^{-h_i-h_j+\frac{1}{3}\sum_{k=1}^{4} h_k}$$

for

$$r(z) = \frac{z_{12}z_{34}}{z_{13}z_{24}} \text{ : crossing ratio.}$$

By conf cov maps

$$(z_1,z_2,z_3,z_4) \mapsto (\underset{\substack{\parallel \\ z}}{r(z)}, 0, \infty, 1)$$

Using OPE we get

$$\langle \Phi_1(z)\,\Phi_2(0)\;\;\Phi_3(\infty)\;\;\Phi_4(1)\rangle$$

$$= \sum_{k\in B_0} C_{12k}\; z^{h_k-h_1-h_2}\left(\langle \Phi_k(z)\,\Phi_3(\infty)\,\Phi_4(1)\rangle + O(z)\right)$$

$$= \sum_{k\in B_0} C_{12k}\,C_{k34}\; z^{h_k-h_1-h_2}\,(1+\underbrace{O(z)}_{\text{contribution of descendants}})$$

$$= \sum_{k} C_{12k}C_{k34}\,\overline{F}^{s}(z)$$

If OPE is associative then

↑ s-channel 4-pt conf block

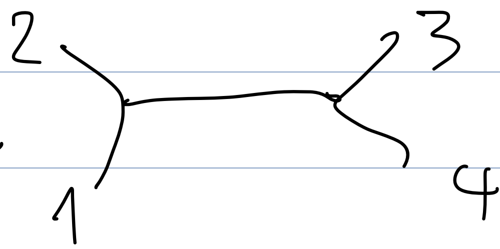
$$\sum_K C_{12K} C_{K34} F^s(z)$$

$$\shortparallel$$

$$\sum_K C_{23K} C_{K41} F^t(z)$$

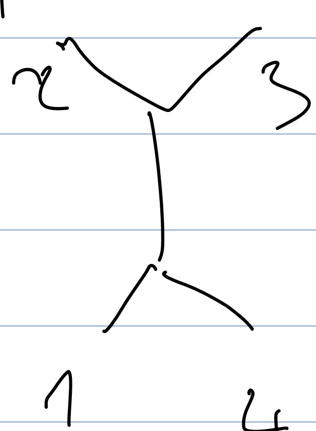
represent using s, t channels then

$$\sum_K C_{12K} C_{K34}$$



$$\shortparallel$$

$$\sum_K C_{23K} C_{K41}$$



this is crossing symmetry

! Using OPE we obtain Clebsch Gordan type result for Vir-rep:

Let $[\Phi_i] \in$ Vir-rep then

$$[\Phi_i] \times [\Phi_j] = \sum_{\ell \in B_1} N_{ij}^{\ell} [\Phi_j]$$

for $N_{jk}^{\ell} =$

$$\left| \{ \Phi_\ell^\alpha \in [\Phi_\ell] \mid \Phi_\ell^\alpha \text{ appears in } O(z) \text{ of } \Phi_i(z)\Phi_j(0) \} \right|$$

$$+ \left(1 - \delta_{C_{ijk,0}}\right)$$

(Fusion rule)